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CONFIDENCE INTERVALS FOR THE RELIABILITY OF MULTI-COMPONENT SYSTEMS

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BALLISTIC RESEARCH LABORATORIES

ABERDEEN PROVING GROUND, MARYLAND

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B A L L I S T I C   R E S E A R C H   L A B O R A T O R I E S

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ABSTRACT

A procedure for finding a confidence interval for the reliability, P, of a multi-component device is presented which utilizes Bernoulli test data pertaining to the component parts.

In particular, n_i Bernoulli trials are carried out for the i^{th} part of a system built up from k different parts. On the basis of the number of observed failures X_i $(i=1,\ldots,k)$ an interval estimate of the reliability (or probability of functioning) of the system is constructed.

In the series case, with k parts, the minimum and maximum of P (as a function of q_i's) is found given the condition that $\sum n_i q_i = \xi$, q_i denoting the probability of the i^{th} part failing in a single trial. The bounds are thus functions of ξ. Since $\sum X_i$ has expectation ξ and is closely approximated by well-known, simple distributions, a confidence interval for ξ, and hence for the bounds of P, and thus for P, can be found.

In the general case, that is, for a system having a mixture of series and parallel connected parts, parametric minima and maxima for P are found using several approaches, and independent confidence intervals are used to construct an interval for P. A comparison of the results using this approach with the results of other procedures is made for some numerical examples.

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INTRODUCTION

The reliability, P, of a device may be defined simply as the probability that it functions correctly under specified conditions. For many devices an interval estimate of this number P is desired. It is sometimes impractical, if not impossible, to test many assembled devices, and the suggestion has been made that parts be tested separately, and the results combined to form an interval estimate. The present report indicates a means of constructing an interval for P with any desired confidence coefficient, preferably, however, greater than .90. As is the case with all non-Neyman-shortest confidence intervals based on discrete variables, the actual probability of coverage will generally be somewhat higher than the selected coefficient.

For any given device with k distinct parts, P may be written in terms of $q_1,\ldots,q_k$, where q_i denotes the probability of the i^{th} part failing. The parts are assumed to function independently of one another within the device in the same way and under the same conditions as when tested separately. The problem considered here is that of finding a confidence interval for P, given that n_i Bernoulli trials have been conducted on the i^{th} part $(i=1,\ldots,k)$ and that X_i failures have been observed in the n_i trials.

The approach of this report is to solve the problem first for the simple series case, and then extend the results to the general case.

For a simple series device consisting of k parts, P (expressed in terms of the q_i) can be minimized and maximized under the restriction $\sum n_i q_i = \xi$. Thus P will be bounded on either side by functions of ξ. Since $X = \sum X_i$ has expectation ξ and is approximately Poisson (or binomially) distributed, one can easily find a confidence interval for ξ based on X, and by appropriate computations, for the two bounding functions of ξ, and hence for P.

The extension from the simple series to the more complicated devices follows from factoring P into its series and non-series parts.

Both parts can be parametrically bounded, the series part as outlined above and the non-series part using several approaches. From these bounds one can write upper and lower bounds for P, and by using several independent confidence intervals, a confidence interval for P may be obtained.

Numerical examples are given which illustrate the procedure and provide comparisons with the results of some other currently used methods.

The reader who wishes to apply the results must be willing to perform some numerical computations. Situations undoubtedly exist for which the present approach is of little use, and in many other cases the experimenter's supply of ingenuity may be heavily taxed to provide shortcuts peculiar to the problem at hand. The present discussion is designed to suggest several solutions to the problem, and to urge the reader to choose the best one available. Most likely this choice cannot be made without first performing trial and error calculations.

PARAMETRIC BOUNDS FOR P IN THE SIMPLE SERIES CASE

A very simple case is a device consisting of k different parts connected in series. Assuming that the parts function independently, the reliability P equals $\prod(1-q_i)$ where q_i equals the probability of the i^{th} part failing. We can find (for reasons which may not yet be obvious) the range of possible values that $\prod(1-q_i)$ can assume given that $\Sigma\, n_i q_i$ equals ξ, where the n_i are positive integers, and ξ is a positive fixed number.

The problem of finding the range of P given ξ can be solved several ways, perhaps most easily by simple trial and error calculations. For known n_i and fixed ξ, the minimum and maximum values of P, expressed in terms of functions of ξ and the n_i, may be found by numerical trial and error of various values of the q_i. When P is near one (say greater than one half), it is clear from the algebraic expansion of $\prod(1-q_i)$ that $1 - \Sigma\, q_i$ will dominate, and it follows that the minimum and maximum of P (at worst, approximately) will be attained when $\Sigma\, q_i$, subject to $\Sigma\, n_i q_i = \xi$, is respectively maximized and minimized.

For example, if $k=3$, $n_1=500$, $n_2=250$, $n_3=300$, $\xi=1.5$, the condition is $500q_1 + 250q_2 + 300q_3 = 1.5$, and $q_1 + q_2 + q_3$ is minimized when $q_1 = 1.5/500$, $q_2 = q_3 = 0$, and maximized when $q_1 = q_3 = 0$, $q_2 = 1.5/250$. A little arithmetic shows that P is indeed maximized and minimized, respectively, at these points. Similar results easily may be seen to hold for all values of ξ less than 250. Hence for any fixed value of ξ in this range,

$$1 - \xi/250 \leq P \leq 1 - \xi/500$$

and P may assume any value in this interval, which has width $\xi/500$.

The general form of the bounds for P can be somewhat complicated, depending on the values of the n_i given. However, the following two general cases are of greatest interest:

A. If $n_i = \overline{n} = \Sigma\, n_i/k$ for $i=1,\ldots,k$,

then $$1 - \xi/\overline{n} \leq P \leq (1 - \xi/\overline{n}k)^k$$

B. If $0 < \sum n_i - k\min_i n_i$ and $0 < \xi < \min(\min_i n_i, \sum n_i - k\min_i n_i)$,

then $$1 - \xi/\min_i n_i \leq P \leq 1 - \xi/\max_i n_i, \text{ where } i=1,\ldots,k.$$

SAMPLING AND USE OF THE BOUNDS

Suppose that the results of n_i Bernoulli trials are known for the i^{th} part of a simple series device. If X_i denotes the number of failures observed, X_i is binomially distributed with parameters n_i and q_i. The variable $X = \Sigma X_i$ does not follow a distribution having a simple form, but it is easy to verify that the expectation and variance of X are ξ and $\xi - \Sigma n_i q_i^2$ respectively. Using the method of Lagrange multipliers, it is also easily shown that given $\xi = \Sigma n_i q_i$, $\Sigma n_i q_i^2$ attains its minimum value when $q_i = \xi / \Sigma n_i$ for all i, and hence the variance is largest when all the q_i's are equal. But under the latter circumstances, X is binomially distributed with parameters Σn_i and $\xi/\Sigma n_i$. If X were Poisson distributed with parameter ξ, then the variance would also be ξ.* Writing the variance of X under the above binomial assumptions as $V_{max}(X)$, and the variance of X under Poisson assumptions as $V_P(X)$, the following inequality holds for all values of n_i and q_i, $i=1,\ldots,k$:

$$V(X) \leq V_{max}(X) \leq V_P(X) \qquad \text{or}$$

$$\xi - \Sigma n_i q_i^2 \leq \xi - \xi^2/\Sigma n_i \leq \xi.$$

Thus referring X to either the Poisson or the appropriate binomial distribution, one should not be surprised to find greater probabilities for the extreme tail values than under the true distribution. If so, when using binomial or Poisson confidence intervals with X, one may expect them to be conservative in the sense that the probability of containing the unknown parameter ξ will be at least as high as the confidence coefficient of the binomial or Poisson intervals. In the next section, comparisons will be presented concerning these probabilities.

For the present, assuming that in referring X to the Poisson or binomial confidence intervals one will not be led astray, it is a

* Were each X_i Poisson, X would also be Poisson distributed.

simple matter to look up one or two-sided confidence limits for P, once the appropriate parametric bounds have been chosen. For example, if the α-level two-sided Poisson or binomial confidence limits for ξ based on X turn out to be t_1, t_2, then whenever $t_1 \leq \xi \leq t_2$, it is also true that (using the case of $n_i = \bar{n}$ here)

$$1 - t_2/\bar{n} \leq 1 - \xi/\bar{n} \leq P \leq (1 - \xi/\bar{n}k)^k \leq (1 - t_1/\bar{n}k)^k$$

and the resulting confidence interval covers P with probability at least $1-\alpha$. There will be a growth in actual confidence coefficient in addition to that due to "too-conservative" intervals because P is a fixed number between two functions of ξ, and the confidence interval covers both of these functions.

In many cases, the q_i will be known to be close to zero, and the values of n_i will be such that ξ may be safely assumed to be within the range of case B of the previous section, or preferably, case A. There is a way of converting all cases into the equal sample size case, which unfortunately throws away part of the available information and subjects the confidence interval for P to additional fluctuation. This will be discussed further in the section on complex systems.

THE BINOMIAL AND POISSON APPROXIMATIONS

In this section the distribution of $X = \Sigma X_i$ is in specific cases compared to the distribution of X when referred to the Poisson or appropriate binomial distribution discussed above. A comparison of the confidence coefficients for ξ under these circumstances is also made.

For the following cases, the density of X has been computed to five decimals:

(1) $n_1 = n_2 = 15$, $n_1q_1 + n_2q_2 = 3$ for $q_1 = 0$, .01, .05, .1 and .2 and the Poisson density with parameter 3.

(2) $n_1 = 5$, $n_2 = 15$, $n_1q_1 + n_2q_2 = 3$ for $q_1 = 0$, .03, .15, .30 and .60.

TABLE I

Density of $X_1 + X_2$ when X_i: $B(n_i, q_i)$; $n_1 = n_2 = 15$, $n_1q_1 + n_2q_2 = 3$ $(q_2 = .2 - q_1)$

	Binomial Assumptions				
X	q_1=0 or .2	q_1=.01 or .19	q_1=.05 or .15	q_1=.1	Poisson density*
0	.03518	.03646	.04047	.04239	.04979
1	.13194	.13380	.13908	.14130	.14936
2	.23090	.23046	.22868	.22760	.22404
3	.25014	.24741	.23952	.23609	.22404
4	.18760	.18542	.17949	.17707	.16803
5	.10318	.10297	.10248	.10230	.10082
6	.04299	.04389	.04635	.04736	.05041
7	.01382	.01468	.01705	.01804	.02160
8	.00346	.00390	.00520	.00576	.00810
9	.00067	.00085	.00133	.00156	.00270
10	.00010	.00014	.00029	.00036	.00081
11	.00001	.00002	.00005	.00007	.00022
12			.00001	.00001	.00006
13					.00001

* This column applies to Table II also, but is not listed there.

TABLE II

Density of $X_1 + X_2$ when X_i: $B(n_i, q_i)$; $n_1 = 5$, $n_2 = 15$,

$n_1q_1 + n_2q_2 = 3$

X	$q_1=0$, $q_2=.2$	$q_1=.03$, $q_2=.19$	Binomial Assumptions $q_1=q_2=.15$	$q_1=.3$, $q_2=.1$	$q_1=.6$, $q_2=0$
0	.03518	.03640	.03876	.03460	.01024
1	.13194	.13371	.13680	.13183	.07680
2	.23090	.23047	.22934	.23200	.23040
3	.25014	.24753	.24283	.25089	.34560
4	.18760	.18554	.18212	.18711	.25920
5	.10318	.10300	.10285	.10240	.07776
6	.04299	.04386	.04537	.04272	
7	.01382	.01463	.01601	.01394	
8	.00346	.00388	.00459	.00362	
9	.00067	.00083	.00108	.00075	
10	.00010	.00014	.00021	.00013	
11	.00001	.00002	.00004	.00002	

In the two tables, $n_1q_1 + n_2q_2$ is always three, and the frequency functions may be compared directly with the approximating binomial and Poisson frequency functions. In Table I, the maximal variance of X occurs when $q_1 = .1$, and in Table II, when $q_1 = .15$.* It will be noted in both tables that for the extreme tail values of X, the approximating binomial and Poisson probabilities are too large. It is also apparent that since the probability that X equals x is a continuous function of q_1, that for all values of q_1 (and hence q_2) in both tables, the binomial and Poisson probabilities will be too large in both tails, and for more values of X in the upper than in the lower tail. Also, in the tables, the binomial and Poisson tail probabilities are greater than the corresponding true probabilities for nearly the same values of X (differing at most by one point), regardless of the value of q_1. The actual probabilities in both tails increase as q_1 and q_2 approach equality, although they are always less than the corresponding Poisson probabilities, suggesting that for arbitrary k, the distribution of X when all the

* This is for the true distribution of $X = X_1 + X_2$.

q_i's are equal is "closer" to the corresponding Poisson distribution than when two q_i's differ. This is not surprising, for in case the q_i's are all equal, X is binomially distributed with parameters Σn_i, $\xi/\Sigma n_i$, and if ξ is small relative to Σn_i, the Poisson approximation to the binomial is known to be very good.

In this optimum case, wherein X is binomially distributed, one may easily compare the tabled binomial and Poisson densities. For a single binomial variable X having parameters n, $\bar{\xi} = \xi/n$, the tail probabilities have been compared with the corresponding tail probabilities of the Poisson distribution (parameter $n\bar{\xi} = \xi$) in the following cases:

(1)	$n\bar{\xi}=1$;	$n=5$, $\bar{\xi}=.2$;	$n=10$, $\bar{\xi}=.1$;	$n=100$, $\bar{\xi}=.01$
(2)	$n\bar{\xi}=10$;	$n=50$, $\bar{\xi}=.2$;	$n=100$, $\bar{\xi}=.1$;	$n=1000$, $\bar{\xi}=.01$
(3)	$n\bar{\xi}=50$;	$n=100$, $\bar{\xi}=.5$;	$n=500$, $\bar{\xi}=.1$;	$n=1000$, $\bar{\xi}=.05$

In all cases, exactly the same remarks as made for the sum of two binomial variables apply, with the obvious additional observation that as n increases, the tail probabilities of the binomial distribution approach the corresponding Poisson probabilities. Both the binomial and Poisson distributions are asymptotically normally distributed (that is, as n and ξ, respectively, increase) and hence if X is large, one may expect the difference between the two sets of confidence limits to be relatively small.

In any particular problem, the preceding indicates that one will achieve narrower confidence bounds for ξ and hence for P, by using the binomial rather than Poisson limits. When ξ is small and Σn_i is relatively large, the difference between the two approximating distributions becomes quite small. When Σn_i is large and X turns out to be small, the binomial confidence limits are easily and quite accurately approximated by the Poisson limits. In the following tables and remarks, comparisons are made of the Poisson and binomial intervals for certain small values of $n = \Sigma n_i$, α, and $X \leq 50$. For large

values of n, the Poisson limits will be much closer to the binomial limits than for those values tabled.

These tables have been drawn from three widely available sources:

Tables III and VI are from the binomial graphs and Poisson confidence limit tables in Biometrika Tables for Statisticians, Volume I, edited by E. S. Pearson and H. O. Hartley. They are based on the equal-tails approach, which chooses, for a given value of X (say c) the values of a Poisson parameter λ, such that, in the two-sided case,

$$\sum_{i=c}^{\infty} e^{-\underline{\lambda}}\, \underline{\lambda}^{i}/i! = \alpha/2 = \sum_{i=0}^{c} e^{-\overline{\lambda}}\, \overline{\lambda}^{i}/i!$$

where $(\underline{\lambda}, \overline{\lambda})$ is to be the α-level confidence interval. The two-sided intervals are tabled for Poisson X = 0(1)30(5)50, α = .10, .05, .02, .01, .002, and binomial $X \leq n$, for $n \leq 1000$, α = .05, .01.

Tables IV and VII are based on tables published by Crow and Gardner in Biometrika, as follows:

(1) Crow, E. L. Confidence Intervals for a Proportion, Biometrika, Volume 43 (1956), pages 423-435. (For binomial $X \leq n$, n = 1(1)30, α = .10, .05, .01)

(2) Crow, E. L. and Gardner, R. S. Confidence Intervals for the Expectation of a Poisson Variable, Biometrika, Volume 46 (1959), pages 441-453. (For X = 0(1)300, α = .20, .10, .05, .01, .001) The system used, described in detail in (1), is optimum in a geometrical sense and generally yields bounds of less width than does the equal-tails method.

Tables V and VIII are based on tables published by Blyth and Hutchinson in Biometrika, as follows:

(1) Tables of Neyman-Shortest Unbiased Confidence Intervals for the Binomial Parameter, Biometrika, Volume 47 (1960), pages 381-391. (For $X \leq n$, n = 2(1)24(2)50, α = .05, .01)

(2) Tables of Neyman-Shortest Unbiased Confidence Intervals for the Poisson Parameter, Biometrika, Volume 48 (1961),

pages 191-194. (For $X = 0(1)250$, $\alpha = .05, .01$)

The optimum property of these intervals may be described as follows: Among all unbiased α-level confidence intervals, the tabled intervals uniformly minimize the probability of covering false values. An unbiased interval A is defined such that if P_θ denotes probability when the distribution parameter is θ, that $P_\theta(\theta \epsilon A) \leq 1 - \alpha$ for all θ, and $P_\theta(\theta' \epsilon A) \leq P_\theta(\theta \epsilon A)$ for all θ, θ'.

In the binomial portions of each table, the entries are nc_1, nc_2, where c_1, c_2 are the tabled α-level, two-sided confidence limits for p.

When using the Neyman-shortest tables, it is necessary to choose a random number between zero and one and add it to X and then read the appropriate entry.

TABLE III

Two-sided Poisson and Binomial* Confidence Limits:

Equal Tails, $\alpha = .05$

X	n=10		n=20		n=30		Poisson	
0	.00	3.1	.00	3.4	.00	3.5	.000	3.69
1	.02	4.4	.02	5.0	.06	5.2	.0253	5.57
2	.25	5.6	.25	6.3	.22	6.6	.242	7.22
3	.68	6.5	.65	7.5	.63	8.0	.619	8.77
4	1.2	7.4	1.2	8.7	1.1	9.3	1.09	10.24
5	1.9	8.1	1.7	9.8	1.7	10.4	1.62	11.67
6	2.6	8.8	2.4	10.8	2.3	11.6	2.20	13.06
7	3.5	9.3	3.0	11.8	3.0	12.7	2.81	14.42
8	4.4	9.8	3.8	12.8	3.6	13.7	3.45	15.76
9	5.6	10.0	4.6	15.7	4.4	14.8	4.12	17.08
10	6.9	10.0	5.4	14.6	5.2	15.8	4.80	18.39
11			6.3	15.4	5.9	16.8	5.49	19.68
12			7.2	16.2	6.8	17.8	6.20	20.96
13			8.2	17.0	7.6	18.8	6.92	22.23
14			9.2	17.6	8.4	19.6	7.65	23.49
15			10.2	18.3	9.4	20.6	8.40	24.74
16			11.3	18.8	10.4	21.6	9.15	25.98
17			12.5	19.4	11.2	22.4	9.90	27.22
18			13.7	19.8	12.2	23.2	10.67	28.45
19			15.0	20.0	13.2	24.1	11.44	29.67
20			16.6	20.0	14.2	24.8	12.22	30.89
21					15.2	25.6	13.00	32.10
22					16.3	26.4	13.79	33.31
23					17.3	27.0	14.58	34.51
24					18.4	27.7	15.38	35.71
25					19.6	28.3	16.18	36.90
26					20.7	28.9	16.98	38.10
27					22.0	29.4	17.79	39.28
28					23.4	29.8	18.61	40.47
29					24.8	29.9	19.42	41.65
30					26.5	30.0	20.24	42.83

* The above binomial confidence limits are for np.

TABLE IV

Two-Sided Limits: Crow and Gardner*, α = .05

X	n=5		n=10		n=20		n=30		Poisson	
0	.000	2.5	.000	2.67	.000	2.86	.000	3.00	.0	3.285
1	.050	3.28	.05	3.97	.06	4.44	.06	4.89	.051	5.323
2	.380	4.06	.37	6.03	.36	5.86	.36	6.15	.355	6.686
3	.945	4.62	.87	6.19	.84	7.02	.84	7.32	.818	8.102
4	1.72	4.95	1.50	7.33	1.42	8.22	1.41	8.76	1.366	9.598
5	2.50	5.00	2.22	7.78	2.08	9.34	2.04	9.72	1.970	11.177
6			2.67	8.50	2.80	10.7	2.73	10.9	2.613	12.817
7			3.81	9.13	2.86	11.8	3.00	12.1	3.285	13.765
8			3.97	9.63	4.18	13.0	3.93	13.2	3.285	14.921
9			6.03	9.95	4.44	14.1	4.89	14.3	4.460	16.768
10			7.33	10.0	5.86	14.1	5.25	15.7	5.323	17.633
11					5.86	15.6	6.15	16.8	5.323	19.050
12					7.02	15.8	7.08	17.9	6.686	20.335
13					8.22	17.1	7.32	19.1	6.686	21.364
14					9.34	17.2	8.76	20.3	8.102	22.945
15					10.7	17.9	9.72	20.3	8.102	23.762
16					11.8	18.6	9.72	21.2	9.598	25.400
17					13.0	19.2	10.9	22.7	9.598	26.306
18					14.1	19.6	12.1	22.9	11.177	27.735
19					15.6	19.9	13.2	23.8	11.177	28.966
20					17.1	20.0	14.3	24.8	12.817	30.017
21							15.7	25.1	12.817	31.675
22							16.8	26.1	13.765	32.277
23							17.9	27.0	14.921	34.048
24							19.1	27.3	14.921	34.665
25							20.3	28.0	16.768	36.030
26							21.2	28.6	16.77	37.67
27							22.7	29.2	17.63	38.16
28							23.8	29.6	19.05	39.76
29							25.1	29.9	19.05	40.94
30							27.0	30.0	20.33	41.75

* The above binomial confidence limits are for np.

TABLE V

Two-Sided Neyman-Shortest Limits:* α = .05

X+Y	n=5		n=10		n=20		n=30		Poisson	
.0	.0	.0	.0	.0	.0	.0	.0	.0	.00	.00
.5	.0	2.2	.0	2.5	.0	2.6	.0	2.7	.0	2.8
1.0	.0	2.6	.0	3.0	.0	3.2	.0	3.3	.0	3.5
1.5	.0	3.4	.0	4.0	.0	4.3	.0	4.4	.0	4.7
2.0	.0	3.8	.1	4.4	.0	4.8	.0	5.1	.0	5.4
2.5	.2	4.3	.1	5.2	.1	5.7	.0	5.8	.1	6.4
3.0	.5	4.5	.4	5.6	.4	6.2	.3	6.6	.4	7.1
3.5	.7	4.8	.6	6.3	.5	7.1	.6	7.4	.5	8.0
4.0	1.2	5.0	1.0	6.6	.8	7.6	.9	7.8	.8	8.6
4.5	1.6	5.0	1.2	7.2	1.1	8.3	1.0	8.6	1.0	9.5
5.0	2.4	5.0	1.7	7.5	1.6	8.8	1.5	9.3	1.4	10.1
5.5			2.0	8.0	1.7	9.3	1.6	9.9	1.6	10.9
6.0			2.5	8.3	2.2	9.8	2.1	10.5	2.0	11.5
6.5			2.8	8.8	2.4	10.4	2.4	11.1	2.2	12.3
7.0			3.4	9.0	3.0	10.8	2.7	11.4	2.6	12.9
7.5			3.7	9.4	3.2	11.4	3.0	12.3	2.9	13.7
8.0			4.4	9.6	3.6	11.8	3.6	12.6	3.3	14.3
8.5			4.8	9.9	4.0	12.4	3.9	13.2	3.6	15.0
9.0			5.6	9.9	4.6	12.8	4.2	13.8	3.9	15.6
9.5			6.0	10.0	4.8	13.4	4.5	14.4	4.2	16.3
10.0			7.0	10.0	5.4	13.8	5.1	14.7	4.6	16.9

* The above binomial confidence limits are for np.

TABLE V

Two-Sided Neyman-Shortest Limits: $\alpha = .05$

(Continued)

X+Y	n=5		n=10		n=20		n=30		Poisson	
11					6.2	14.6	6.0	15.9	5.3	18.2
12					7.2	15.4	6.6	16.8	6.0	19.5
13					8.2	16.4	7.5	18.0	6.8	20.8
14					9.2	17.0	8.4	18.9	7.5	22.1
15					10.2	17.8	9.3	19.8	8.2	23.3
16					11.2	18.4	10.2	20.7	9.0	24.6
17					12.4	19.2	11.1	21.6	9.7	25.8
18					13.8	19.6	12.0	22.5	10.5	27.1
19					15.2	20.0	13.2	23.4	11.3	28.3
20					16.8	20.0	14.1	24.0	12.0	29.5
21							15.3	24.9	12.8	30.7
22							16.2	25.8	13.6	31.9
23							17.4	26.4	14.4	33.1
24							18.6	27.3	15.2	34.3
25							19.5	27.9	16.0	35.5
26							20.7	28.5	16.8	36.7
27							22.2	29.1	17.6	37.9
28							23.4	29.7	18.4	39.1
29							24.9	30.0	19.3	40.3
30							26.7	30.0	20.1	41.5

The above binomial confidence limits are for np.

TABLE VI

Two-Sided Equal-Tails Limits*: $\alpha = .01$

X	n=10		n=20		n=30		Poisson	
0	.00	4.1	.00	4.7	.00	4.9	.000	5.30
1	.00	5.4	.00	6.2	.00	6.6	.00501	7.43
2	.12	6.5	.12	7.8	.08	8.2	.103	9.27
3	.37	7.4	.40	8.9	.38	9.6	.338	10.98
4	.77	8.1	.74	10.1	.75	10.9	.672	12.59
5	1.3	8.7	1.1	11.2	1.1	12.2	1.08	14.15
6	1.9	9.2	1.7	12.2	1.6	13.3	1.54	15.66
7	2.6	9.6	2.3	13.1	2.5	14.4	2.04	17.13
8	3.5	9.9	2.9	14.0	2.8	15.4	2.57	18.58
9	4.6	10.0	3.6	14.8	3.4	16.5	3.13	20.00
10	5.9	10.0	4.4	15.7	4.1	17.5	3.72	21.40
11			5.2	16.4	4.7	18.4	4.32	22.78
12			6.0	17.1	5.6	19.4	4.94	24.14
13			6.9	17.7	6.3	20.2	5.58	25.50
14			7.8	18.3	7.1	21.2	6.23	26.84
15			8.8	18.9	8.0	22.1	6.89	28.16
16			9.9	19.3	8.8	22.9	7.57	29.48
17			11.1	19.6	9.8	23.7	8.25	30.79
18			12.2	19.9	10.6	24.4	8.94	32.09
19			13.8	20.0	11.6	25.3	9.64	33.38
20			15.3	20.0	12.5	25.9	10.35	34.67
21					13.5	26.6	11.07	35.95
22					14.6	27.2	11.79	37.22
23					15.6	27.4	12.52	38.48
24					16.7	28.4	13.25	39.74
25					17.8	28.9	14.00	41.00
26					19.1	29.2	14.74	42.25
27					20.4	29.6	15.49	43.50
28					21.8	29.9	16.24	44.74
29					23.4	30.0	17.00	45.98
30					25.1	30.0	17.77	47.21

* The above binomial confidence limits are for np.

TABLE VII

Two-Sided Limits: Crow and Gardner,* α = .01

X	n=5		n=10		n=20		n=30		Poisson	
0	.000	3.01	.00	3.12	.00	4.18	.00	4.53	.000	4.771
1	.010	3.89	.01	5.12	.02	5.86	.00	6.18	.010	6.914
2	.165	4.47	.16	6.24	.16	7.50	.15	7.68	.149	8.727
3	.530	4.84	.48	7.03	.46	8.48	$.45^-$	9.30	.436	10.473
4	1.11	4.99	.93	7.82	.88	10.0	.84	10.4	.823	12.347
5	1.99	5.00	1.50	8.50	1.38	11.5	1.35^-	11.6	1.279	13.793
6			2.18	9.07	1.96	12.0	1.89	12.9	1.785	15.277
7			2.97	9.52	2.58	12.7	2.49	14.1	2.330	16.801
8			3.76	9.84	3.26	14.1	3.12	15.2	2.906	18.362
9			4.88	9.99	4.00	14.5	3.81	16.1	3.507	19.462
10			6.24	10.0	4.18	15.8	4.53	17.1	4.130	20.676
11					5.48	16.0	4.53	18.4	4.771	22.042
12					5.86	16.7	5.94	19.6	4.771	23.765
13					7.26	17.4	6.18	20.1	5.829	24.925
14					7.98	18.0	7.47	20.8	6.668	25.992
15					8.48	18.6	7.68	22.3	6.914	27.718
16					10.0	19.1	9.24	22.5	7.756	28.852
17					11.5	19.5	9.87	23.8	8.727	29.900
18					12.5^+	19.8	10.4	24.1	8.727	31.839
19					14.1	20.0	11.6	25.5	10.009	32.547
20					15.8	20.0	12.9	25.5	10.473	34.183
21							13.9	26.2	11.242	35.204
22							14.8	26.9	12.347	36.544
23							15.9	27.5	12.347	37.819
24							17.1	28.1	13.793	38.939
25							18.4	28.6	13.793	40.373
26							19.6	29.2	15.28	41.39
27							20.7	29.6	15.28	42.85
28							22.3	29.8	16.80	43.91
29							23.8	30.0	16.80	45.26
30							25.5	30.0	18.36	46.50

* The above binomial confidence limits are for np.

TABLE VIII

Two-Sided Neyman-Shortest Limits*: α = .01

X+Y	n=5		n=10		n=20		n=30		Poisson	
.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0
.5	.0	3.1	.0	3.6	.0	4.0	.0	4.2	.0	4.4
1.0	.0	3.4	.0	4.1	.0	4.6	.0	4.8	.0	5.1
1.5	.0	4.1	.0	5.1	.0	5.7	.0	6.2	.0	6.6
2.0	.0	4.3	.0	5.5	.0	6.2	.0	6.6	.0	7.2
2.5	.0	4.7	.0	6.2	.0	7.3	.0	7.6	.0	8.4
3.0	.2	4.8	.2	6.6	.2	7.8	.3	8.1	.1	9.1
3.5	.3	5.0	.2	7.2	.2	8.5	.3	9.2	.2	10.2
4.0	.7	5.0	.5	7.5	.4	9.0	.6	9.6	.4	10.8
4.5	.9	5.0	.7	8.1	.6	9.7	.6	10.4	.5	11.8
5.0	1.6	5.0	1.1	8.3	1.0	10.2	.9	10.8	.8	12.4
5.5			1.2	8.8	1.1	10.9	1.0	11.6	1.0	13.4
6.0			1.7	8.9	1.4	11.2	1.5	12.0	1.3	14.0
6.5			1.9	9.3	1.6	11.8	1.5	12.9	1.4	14.9
7.0			2.5	9.5	2.0	12.2	2.1	13.2	1.8	15.5
7.5			2.8	9.8	2.2	12.8	2.1	14.1	2.0	16.4
8.0			3.4	9.8	2.8	13.2	2.7	14.4	2.3	17.0
8.5			3.8	10.0	3.0	13.8	2.7	15.0	2.5	17.8
9.0			4.5	10.0	3.4	14.0	3.3	15.6	2.9	18.4
9.5			4.9	10.0	3.8	14.6	3.6	16.2	3.1	19.2
10.0			5.9	10.0	4.2	15.0	3.9	16.5	3.5	19.8

* The above binomial confidence limits are for np.

TABLE VIII

Two-Sided Neyman-Shortest Limits: $\alpha = .01$

(Continued)

X+Y	n=5		n=10		n=20		n=30		Poisson	
11					5.0	15.8	4.8	17.4	4.1	21.2
12					6.0	16.6	5.4	18.6	4.8	22.6
13					6.8	17.2	6.3	19.5	5.4	24.0
14					7.6	18.0	6.9	20.4	6.0	25.3
15					8.8	18.6	7.8	21.3	6.7	26.7
16					9.8	19.0	8.7	22.2	7.4	28.0
17					11.0	19.6	9.6	23.1	8.1	29.3
18					12.2	19.8	10.5	23.7	8.8	30.6
19					13.8	20.0	11.4	24.6	9.5	31.9
20					15.4	20.0	12.6	25.2	10.2	33.2
21							13.5	26.1	10.9	34.5
22							14.4	26.7	11.6	35.8
23							15.6	27.3	12.4	37.0
24							16.8	27.9	13.1	38.3
25							18.0	28.5	13.8	39.6
26							19.2	29.1	14.6	40.8
27							20.4	29.4	15.3	42.1
28							21.9	29.7	16.1	43.3
29							23.4	30.0	16.8	44.6
30							25.2	30.0	17.6	45.8

It is apparent from visually comparing the Poisson limits with the corresponding binomial limits, that except for a few cases (a total of 16), when $\alpha = .05$ or $.01$, the former completely cover the binomial limits for n = 10, 20, 30 and in four of the tables, for n = 5. In Tables III and VI the only discrepancies occur for $X \leq 2$, where the graphed values, multiplied by n, may be necessarily too imprecise to compare meaningfully with the Poisson readings. In Tables IV and VIII, for the discrepancies occurring when $X \leq 2$, again impreciseness of the binomial tables may be responsible. For the larger values of X the behavior of both intervals is somewhat erratic, and this plus the fact that both the tabled Poisson and binomial intervals generally cover the unknown parameters with probability greater than .95 or .99, indicates that an occasional slight non-inclusion for a particular value of X is not serious. The procedure of referring to the binomial or Poisson limits will still be expected to yield confidence coefficient very near to, if not greater than $1 - \alpha$.

In all the tables, as n increases, the end-points uniformly expand towards the corresponding values in the Poisson case. For $\alpha = .05$ or .01, the fit generally seems to be closer for the Neyman-shortest intervals, which is not surprising since it is the only one of the three methods that exactly attains its confidence coefficient for all parameter values.

For values of X and n other than those tabled, several approximations to the binomial and Poisson distributions exist, a few of which appear in <u>Biometrika Tables for Statisticians</u>.

As a check on the inclusion properties of the binomial and Poisson approximations one may for specific examples, involving known parameters, compute the true probabilities of covering P using the appropriate binomial or Poisson confidence limits (α-level). If the Poisson limits include the binomial limits, which in turn are "too wide" for P, then one would expect in most cases that the Poisson intervals would have the highest confidence coefficient, followed by the "binomial"

confidence coefficient, both of which would be expected to be greater than $1 - \alpha$. Calculations, based on the distribution of X under the various values of q_i appearing in Tables I and II, were made to obtain the probabilities that the Poisson limits and the appropriate binomial limits (α-level) cover $n_1q_1 + n_2q_2$. These probabilities are given in Tables IX and X.

TABLE IX

Probability of the Poisson Limits Covering $n_1q_1 + n_2q_2 = 5$

$n_1 = n_2 = 15 \quad \alpha = .05$

Interval	$q_1 = 0$	$q_1 = .01$	$q_1 = .05$	$q_1 = .10$	Poisson
Equal-Tails	.99575	.99509	.99312	.99221	.98809
Crow and Gardner	.98193	.98041	.97607	.97417	.96649
Neyman-Shortest	.970	.969	.964	.961	.952 (.95)

$n_1 = n_2 = 15 \quad \alpha = .01$

Interval	$q_1 = 0$	$q_1 = .01$	$q_1 = .05$	$q_1 = .10$	Poisson
Equal-Tails	.99921	.99899	.99832	.99797	.99619
Crow and Gardner	.99921	.99899	.99832	.99797	.99619
Neyman-Shortest	.994	.994	.993	.992	.990 (.99)

$n_1 = 5, n_2 = 15 \quad \alpha = .05$

Interval	$q_1 = 0$	$q_1 = .03$	$q_1 = .15$	$q_1 = .30$	$q_1 = .60$
Equal-Tails	.99575	.99514	.99408	.99549	1.00000
Crow and Gardner	.98193	.98051	.97807	.98155	1.00000
Neyman-Shortest	.970	.969	.966	.971	.994

$n_1 = 5, n_2 = 15 \quad \alpha = .01$

Interval	$q_1 = 0$	$q_1 = .03$	$q_1 = .15$	$q_1 = .30$	$q_1 = .60$
Equal-Tails	.99921	.99902	.99867	.99911	1.00000
Crow and Gardner	.99921	.99902	.99867	.99911	1.00000
Neyman-Shortest	.994	.994	.994	.994	.999

TABLE X

Probability of the Binomial Limits Covering $n_1 q_1 + n_2 q_2 = 3$

$n_1 = n_2 = 15 \quad \alpha = .05$

Interval	$q_1 = 0$	$q_1 = .01$	$q_1 = .05$	$q_1 = .10$
Equal-Tails	.99575	.99509	.99312	.99221
Crow and Gardner	.99575	.99509	.99312	.99221
Neyman-Shortest	.962	.960	.954	.951 (.95)

$n_1 = n_2 = 15 \quad \alpha = .01$

Interval	$q_1 = 0$	$q_1 - .01$	$q_1 = .05$	$q_1 = .10$
Equal-Tails	.99921	.99899	.99832	.99797
Crow and Gardner	.99575	.99509	.99312	.99221
Neyman-Shortest	.993	.993	.991	.990 (.99)

$n_1 = 5, n_2 = 15 \quad \alpha = .05$

Interval	$q_1 = 0$	$q_1 = .03$	$q_1 = .15$	$q_1 = .30$	$q_1 = .60$
Equal-Tails	.99575	.99514	.99408	.99549	1.00000
Crow and Gardner	.96057	.95874	.95532	.96089	.98976
Neyman-Shortest	.956	.953	.949(.95)	.956	.992

$n_1 = 5, n_2 = 15 \quad \alpha = .01$

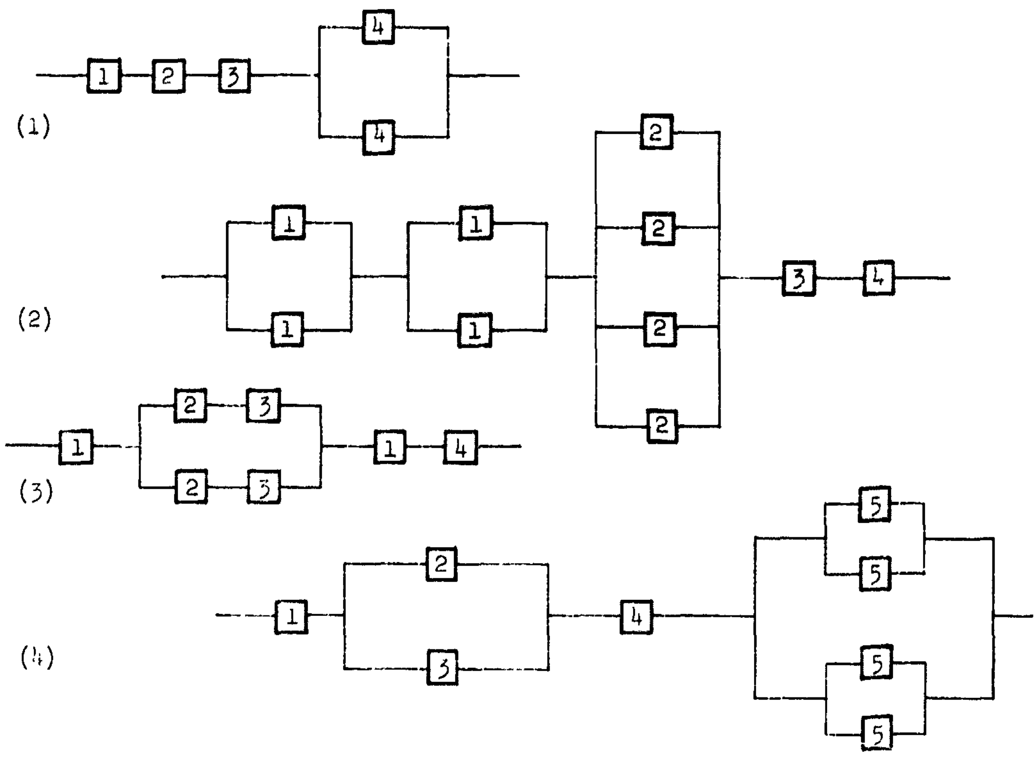
Interval	$q_1 = 0$	$q_1 = .03$	$q_1 = .15$	$q_1 = .30$	$q_1 = .60$
Equal-Tails	.99921	.99902	.99867	.99911	1.00000
Crow and Gardner	.99575	.99514	.99408	.99549	1.00000
Neyman-Shortest	.991	.991	.990(.99)	.991	.998

Again, one may notice that the smallest probabilities occur when $q_1 = q_2 = .1$ and $q_1 = q_2 = .15$, where the variance of X is largest. Also, the Neyman-shortest intervals have the smallest probabilities of all three systems, in most cases, which is due mainly to the exactness of these intervals. Owing to the number of significant decimals published in tables of the latter intervals, the probabilities of coverage may be in error by a few places in the last decimal. For example, in the Poisson case, for $\alpha = .05$, $n_1 = n_2 = 15$, the computed probability using linear interpolation turns out to be .952 whereas the true probability is exactly .95. The corresponding probability when $\alpha = .01$ is correct to three decimals.

When confronted with a value of X resulting from lengthy experimental testing, the procedure adopted might be to select from the available tables the limits which are narrowest for the particular value of X observed, and not to choose the bounds with respect to any optimal mathematical property. Such a procedure defines a kind of composite interval about which little may confidently be said concerning the probability of coverage. Preferably, a single type of interval will be decided upon in advance of the experiment, or else the results using different limits will be presented.

EXTENSION TO COMPLEX SYSTEMS

Unfortunately, the case of a system consisting solely of non-identical parts in series is relatively rare. More often one is interested in systems having a mixture of series and parallel-connected parts. The following diagrams illustrate some of the possibilities:



The squares containing numbers denote parts of the i^{th} type, and natural subdivisions within systems may be called components. For example, the first figure represents a system of three parts in series and one component of two identical parts connected in parallel.

In the following examples the methods used are those suggested for use in more general models.

In the first illustration, suppose that for the i^{th} part, n trials have been conducted. Then from the section on parametric bounds,

$$P = \prod_{i=1}^{3}(1-q_i)(1-q_4^2), \qquad \xi = \sum_{i=1}^{3} n_i q_i, \qquad \text{and}$$

$$(1-q_4^2)(1-\xi/n) \leq P \leq (1-\xi/3n)^3(1-q_4^2).$$

Assuming that the numbers X_i are available, it is easy to find two-sided α-level bounds for ξ, and β-level bounds for q_4 based on $X_1 + X_2 + X_3$ and X_4 respectively. These bounds are such that the events, say $c_1 \leq \xi \leq c_2$, $d_1 \leq q_4 \leq d_2$ independently occur with probabilities $1-\alpha$ and $1-\beta$ respectively. But if both occur at the same time,

$$1-c_2/n \leq \prod_{i=1}^{3}(1-q_i) \leq (1-c_1/3n)^3, \text{ and } 1-d_2^2 \leq 1-q_4^2 \leq 1-d_1^2$$

and hence

$$(1-d_2^2)(1-c_2/n) \leq P \leq (1-c_1/3n)^3(1-d_1^2)$$

with probability $(1-\alpha)(1-\beta)$. If β is chosen small enough, then

$$(1-d_2^2)(1-c_2/n), \ (1-c_1/3n)^3(1-d_1^2)$$

will form a confidence interval for P with coefficient very nearly $1-\alpha$. As β approaches zero, the interval d_1, d_2 will of course widen. But $1-d_1^2$, $1-d_2^2$ will change by a relatively small amount. Hence one may choose β very near to zero, and the resulting confidence interval for P will not be appreciably widened.

It may be noted that one may also lower α a bit in order not to make the bounds so wide, at the expense of a slight widening of the entire interval, to ensure the desired confidence coefficient. One is really free to choose α and β just so long as $(1-\alpha)(1-\beta)$ attains the

desired number. In this example, the width will be found to suffer least if α is lowered as little as possible, and β is made nearly zero. If $(1-\alpha)(1-\beta)$ is slightly below the desired coefficient, the probability of coverage will quite likely be higher, due to the raising of the actual confidence coefficient noted previously.

For the second illustration,

$$P = \prod_{i=3}^{4}(1-q_i)(1-q_2^{4})(1-q_1^{2})^2.$$

Since q_2^4 will generally be very small, one may be willing to ignore it altogether and consider

$$P = \prod_{i=3}^{4}(1-q_i)(1-q_1^{2})^2,$$

in which case the parametric bounds are

$$(1-q_1^{2})^2(1-\xi/\min_{i=3,4} n_i) \leq P \leq (1-\xi/\max_{i=3,4} n_i)(1-q_1^{2})^2$$

(assuming n_i observations for the i^{th} part have been taken, and say,

$$\xi < \sum_{i=3}^{4} n_i - 2\min_{i=3,4} n_i \text{ and } \min_{i=3,4} n_i, \text{ where } \xi = n_3q_3 + n_4q_4).$$

If c_1, c_2 and d_1, d_2 are α and β-level two-sided confidence limits for ξ and q_1 respectively, then

$$(1-d_2^{2})^2(1-c_2/\min n_i),\ (1-c_1/\max n_i)(1-d_1^{2})^2$$

are two-sided confidence limits for P with coefficient greater than or equal to $(1-\alpha)(1-\beta)$. If one wishes to take account of q_2, then γ-level limits b_1, b_2 for q_2 lead to

$$(1-b_2^{4})(1-d_2^{2})^2(1-c_2/\min n_i),\ (1-c_1/\min n_i)(1-d_1^{2})^2(1-b_1^{1})$$

as the limits for P, with confidence coefficient at least $(1-\alpha)(1-\beta)(1-\gamma)$, where γ is chosen to be very nearly zero.

In the third illustration,

$$P = (1-q_1)^2(1-q_4)(1- [1- (1-q_2)(1-q_3)]^2).$$

In this case there are several directions in which one can proceed,

some of which are listed below:

A. One can use the rather primitive inequality

$$(1-q_1)^2(1-q_2^2)(1-q_3)(1-q_4) \leq P \leq (1-q_1)^2(1-q_2^2)(1-q_3^2)(1-q_4)$$

and find four independent confidence limits at low enough levels so that the product of the four confidence coefficients equals $1-\alpha$. The limits will vary considerably in width according to the size of the X_i, n_i, and the levels selected. If the q_i happen to be very close to zero, this may provide narrow limits.

B. If for example

$$1-\xi/\min_{i=2,3} n_i \leq (1-q_2)(1-q_3) \leq 1-\xi/\max_{i=2,3} n_i, \text{ that is}$$

$$0 \leq \xi < \min\left(\min_{i=2,3} n_i, \sum_{i=2}^{3} n_i - 2\min_{i=2,3} n_i\right), \text{ where } \xi = n_2q_2 + n_3q_3,$$

then it follows that

$$(1-q_1)^2(1-q_4)(1-[\xi/\min_{i=2,3} n_i]^2) \leq P \leq (1-[\xi/\max_{i=2,3} n_i]^2)(1-q_1)^2(1-q_4).$$

Using three independent confidence limits, one will in general be able to arrive at narrower confidence bounds for P than by using procedure A.

C. If one is willing to ignore terms in the expansion of P which involve products of three or more q_i, then one may write

$$P = 1 - (2q_1+q_4) + q_1^2 + 2q_1q_4 - q_2^2 - 2q_2q_3 - q_3^2$$

and hence

$$1 - (2q_1+q_4) - (q_2+q_3)^2 \leq P \leq 1 - (2q_1+q_4) + q_1^2 + 2q_1q_4.$$

With arbitrary n_i one will have difficulty finding confidence limits for $(2q_1+q_4)$ and (q_2+q_3). However in the special case that $n_1 = 2n_4$ and $n_2 = n_3$, the problem can be given a simple solution. For in this case one may refer $X_1 + X_4$ and $X_2 + X_3$ to the binomial or Poisson confidence intervals, using the following argument: X_1 is distributed binomially with parameters n_1 and q_1 and variance $n_1q_1 - n_1q_1^2$. If

one refers X_1 to the binomial distribution with parameters $n_1/2$ and $2q_1$, which has variance $n_1q_1 - 2n_1q_1^2 \leq n_1q_1 - n_1q_1^2$, one will be making a conservative approximation in the sense that (as in the case of the Poisson and binomial approximations to ' the tail probabilities will really be smaller than under the approximation. Thus a confidence interval for $2q_1$ based on the approximation will be expected to be too wide. $X_1 + X_4$ may be referred to as binomial (parameters $n_1/2 + n_2$, $\xi/[n_1/2 + n_2]$) or Poisson (parameter $\xi = (n_1/2)(2q_1) + n_2q_2$) for a confidence interval for ξ. But $n_1 = 2n_4$ means that from any confidence interval for ξ one also derives one for $2q_1 + q_4$ since $\xi = n_4(2q_1 + q_4)$ and one may divide the end points through by n_4. Similarly for $q_2 + q_3$, one may derive any level confidence interval desired. If c_1, c_2 are two-sided α-level bounds for $2q_1 + q_4$, and d_1, d_2 are two-sided β-level bounds for $q_2 + q_3$, then the event

$$c_1 \leq 2q_1+q_4 \leq c_2 \text{ and } d_1 \leq q_2+q_3 \leq d_2$$

implies

$$-(q_2+q_3)^2 \geq -d_2^2,\ q_1^2+2q_1q_4 \leq (2q_1q_4)^2/2 \leq c_2^2/2, \text{ and}$$
$$1-c_2 \leq 1-(2q_1+q_4) \leq 1-c_1,$$

and thus

$$1-c_2-d_2^2 \leq P \leq 1-c_1+c_2^2/2$$

occurs with probability greater than or equal to $(1-\alpha)(1-\beta)$. This procedure also will generally yield shorter intervals than procedure A.

As an illustration of how $B(n_1,q_1)$ compares with $B(n_1/2, 2q_1)$, the following frequencies have been tabled:

X	B(100, .02)	B(50, .04)	X	B(100, .02)	B(50, .04)
0	.13262	.12989	6	.01142	.01080
1	.27065	.27059	7	.00313	.00283
2	.27342	.27623	8	.00074	.00063
3	.18227	.18416	9	.00016	.00013
4	.09021	.09016	10	.00002	.00002
5	.03535	.03456	11	.00001	.00000

D. Using the condition

$$\sum_{i=1}^{4} n_i q_i = \xi,$$

one can find the minimum and maximum values of P. These will be functions of ξ, and using X one can find confidence limits for these functions, and hence for P.

Unfortunately, there are several reasons why this procedure is unsatisfactory: First, the bounds are not simple to derive in many cases; and second, their form depends upon the particular range of values within which ξ happens to lie, and in instances this may be quite uncertain; and third, the resultant parametric interval is wider than under alternative procedures.

In general, one can often find bounds for P which are almost as narrow, and in many cases narrower, by considering groups of functions of the q_i such as illustrated in procedure B above, which are easier to derive. Terms of the form $\prod(1-q_i)$ are often the main contributors to the value of P, while the remaining terms may be roughly bounded with little cost in terms of width of the confidence interval. As to which of two or more groups of parameters deserve the smallest confidence level, trial and error calculations may best provide the answer.

One can see that with computation, values of n_i may be chosen to minimize the length of the interval. If one also knows the approximate size of the q_i, a better choice of testing procedure may be made.

In the fourth illustration,

$$P = \prod_{i=1,4}(1-q_i)(1-q_2 q_3)(1-q_3^{\,4}).$$

Usually $q_3^{\,4}$ will be so small as to be entirely negligible, and P may be taken to be

$$\prod_{i=1,4}(1-q_i)(1-q_2 q_3).$$

If one tries to minimize and maximize the entire expression, subject to the restraint, the result, even when all the n_i are equal to $\bar{n}$, is in many situations unsatisfactory. The bounds for one range of

values of $\xi = n_1q_1 + n_4q_4$ are $\xi/\bar{n}$, 1, which, if $\bar{n}$ is large relative to ξ, is much too wide to be of any practical value. In the general case, however, one can often use the bounds for $(1-q_1)(1-q_4)$ and q_2q_3 to provide bounds for P. For example, if $\xi = n_1q_1 + n_4q_4$, $\xi = n_2q_2 + n_3q_3$,

$$(1-\xi_1/\min_{i=1,4} n_i)(1-\xi_2^2/4n_2n_3) \leq P \leq (1-\xi_1/\max_{i=1,4} n_i)$$

is valid when

$$\xi_1 < \min\left(\sum_{i=1,4} n_i - 2\min_{i=1,4} n_i,\ \min_{i=1,4} n_i\right), \text{ and } \xi_2 < \min(n_2, n_3).$$

$X_1 + X_4$ gives bounds for ξ_1 and $X_2 + X_3$ gives bounds for ξ_2, where the product of the two confidence coefficients is $1-\alpha$. In this case a narrower interval will result if the confidence coefficient for ξ_1 is taken much closer to $1-\alpha$ than the coefficient for ξ_2.

In summary, to find a confidence interval for the reliability of a given system, the following approaches are suggested. Some numerical trial and error may be necessary to select the most promising parametric interval.

(1) The upper and lower bounds for P are found under the restriction $\sum n_iq_i = \xi$. The upper bound for one range of values of ξ will be the solutions of the equations $\partial P/\partial q_i = \lambda n_i$ $(i=1,\ldots,k)$, $\lambda \neq 0$, and $\xi = \sum n_iq_i$. The bounds for other ranges of ξ may be found from trial and error of various numerical quantities in P. The lower and upper bounds will both be functions of ξ and the known constants. A confidence interval for ξ based on $\sum X_i$ can by appropriate algebraic manipulation determine the confidence interval for P.

(2) Alternatively, P is separated into products of simpler functions, such as

$$\left[\prod_i (1-q_i)\right]\left[\prod_j (1-q_j^2)\right],$$

and the parametric bounds for each product found, as in illustrations 3 and 4. Then by trial and error the confidence coefficients of limits for the bounds of each product are determined so that the probability

of covering P will be at least $1-\alpha$, and the interval will have small, if not minimum width. If there exists prior information that certain q_i's are small, then one may be willing to neglect powers of q_i greater than two, simplifying the computations. With a small amount of trial and error and even a vague idea about the size of the q_i, one may determine the various confidence coefficients in the product.

(3) If both (1) and (2) are unsatisfactory approaches, one may expand P in terms of the q_i, which may then be divided into homogeneous polynomials of ascending dimension. The first group will be a linear combination of the q_i, the second will be a quadratic form, etc. Often the q_i may be felt to be so small that forms of higher order can safely be neglected. At any rate, the linear group will contribute most to the value of P, and by the device illustrated in method C of the third illustration, one may obtain bounds for this group, and rougher bounds for the other groups, and hence for P. In this case one must have at least some of the n_i in certain known ratios to each other. If the experiment has already been conducted and this is not the case, one may pick a new set of n_i^* such that the n_i^* are properly related to each other and $n_i^* \leq n_i$ for all i. Then if one randomly selects n_i^* observations from the n_i Bernoulli observations previously obtained using a table of random numbers, the new total number of failures in n_i^*, say X_i^*, will be distributed binomially with parameters n_i^* and q_i. One may then proceed as before, using the approach of method C, illustration 3, having thrown away some of the available information, however. This method is a variation of an approach to the confidence interval problem, due to Dr. D. H. Evans of Bell Telephone Laboratories, appearing in Reference 2.

(4) If none of the previous attempts succeeds, one may search for weaker bounds for P and use independent confidence intervals, such as described in method A of the third illustration.

Prior investigation into the desirable sizes of the n_i in general will often simplify computations, in addition to giving a narrower

confidence interval. All the remarks and examples given above also apply to the case in which one-sided intervals are desired. The lower parametric bound of P is usually easy to determine from a small amount of numerical trial and error.

NUMERICAL EXAMPLES

In the following computations the Poisson confidence limits were used in preference to the narrower binomial intervals for the following reasons: (1) For sample sizes as large as in the examples, there is little difference between the two; (2) The binomial limits are not easy to compute, it being necessary to use approximating formulas for the incomplete beta function in some cases; and (3) The Poisson limits are immediately available.

In problems where the sample size is, say, less than 50, it is advisable to use binomial limits.

Consider the first illustration in the preceding section, and suppose that $n_1 = n_2 = n_3 = n_4 = 500$, and the numbers of failures observed are $X_1 = 3$, $X_2 = 1$, $X_3 = 10$, and $X_4 = 2$, and hence the sum over all X_i equals 16. If one fixes

$$\xi = \sum_{i=1}^{4} n_i q_i$$

for various values and investigates the behavior of P, it may be seen that for $\xi < 500$,

$$1-\xi/500 \leq P \leq 1-(\xi/500)^2, \text{ where } P = \prod_{i=1}^{3}(1-q_i)(1-q_4{}^2).$$

In this case, the .05-level two-sided Poisson confidence bounds for 500ξ are 9.598, 25.400 (Crow and Gardner). Accordingly, the confidence bounds for P are .9492, .9996.

Alternatively, one may make use of the inequality

$$(1-q_4{}^2)(1-\xi/500) \leq P \leq (1-\xi/1500)^3(1-q_4{}^2)$$

where

$$\xi = \sum_{i=1}^{3} n_i q_i .$$

The .05-level two-sided confidence bounds for ξ are 8.102, 22.945 (Crow and Gardner) and the $.0^4 1$-level two-sided interval for q_4 (equal-tails Poisson, using the χ^2 distribution as tabled in _Biometrika Tables for Statisticians_) is ($.0^5 6$, .0345). The resulting

.05-level interval for P is (.9530, .9839), appreciably narrower than before.

A third procedure in this illustration involves expanding P in terms of the q_i and ignoring the non-quadratic and linear parts. The result is that

$$P = 1 - \xi + \sum_{i<j} q_i q_j - q_4^2,$$

and maximizing

$$\sum_{i<j} q_i q_j \text{ subject to } 500 \sum_{i=1}^{3} q_i = \xi,$$

one arrives at the inequality

$$1-\xi/500-q_4^2 \leq P \leq 1-\xi/500+\xi^2/3(500)^2.$$

In this procedure the one-sided $.0^4$1-level upper bound for q_4 may be used with the two-sided .05-level bounds for ξ. The bounds for ξ are again 8.102, 22.945, and the upper bound for q_4 is .0335 (using the χ^2 tables again). Performing the calculations, the .05-level bounds for P under the above assumptions are .9529, .9845.

From this example, it is again shown that the method of minimizing the entire expression $P(q_1,\ldots,q_k)$ subject to $\sum n_i q_i = \xi$ does not necessarily result in the shortest confidence intervals. For a particular set of n_i values it will be advantageous to gain an idea of how wide the resulting parametric interval will be under more than one of the approaches suggested. There is nothing "illegal" about picking the method which yields the narrowest parametric interval for a particular problem.

There have been various procedures suggested for finding confidence intervals for P, two of which have appeared in Reference 2. The first method described in the report depends on the Tchebycheff Inequality, and for the previous values of X_i, n_i, yields two-sided .05-level bounds of .9393, 1.0000. Assuming that the mode equals the mean, the interval becomes .9503, .9940. The second method, due to D. H. Evans, depends on a randomization procedure and in the

above illustration necessitates using only 250 observations for each of the first three components. The number of failures Y out of 250 complete systems constructed ranges between 0 and 16. For a few of the values of Y (the expected value of Y is approximately 7) the intervals are as follows:

	Y	Interval Bounds (Poisson; Crow and Gardner)	
α = .05	3	.968	.997
	7	.945	.987
	10	.930	.979
	15	.905	.968

In comparing these results with the previous Poisson limits, it is to be noted that the randomization approach suffers from the disadvantage of being unable to utilize all the information available in the samples. On the other hand, if $n_1 = n_2 = n_3 = 500$ and $n_4 = 1000$, one may obtain a more meaningful comparison of the results of the two procedures. In this case, Y may vary from 10 to 15, and gives the following table of intervals:

TABLE XI

	Y	P(Y)	Confidence Interval (Poisson; Crow and Gardner)		Length
	11	.0000			
	12	.0024	.9593	.9866	.0273
α = .05	13	.0808	.9573	.9866	.0293
	14	.9158	.9541	.9838	.0297
	15	.0009	.9525	.9838	.0313
				Expected Length	.0297

The previous Poisson limits, using the bounds

$$(1-q_4^{2})(1-\xi/500) \leq P \leq (1-\xi/1500)^3(1-q_4^{2})$$

give the .05-level interval (.9538, .9839), with length .0301. While the randomization procedure almost always yields a slightly shorter interval (with probability .9991), the Poisson interval is for any given data a fixed interval. Only the interval corresponding to the

most probable value of Y is completely included in the Poisson interval in this example.

For more complicated models it becomes difficult to calculate the probability density of Y even when the most favorable sample size relations are attained, and hence to compare the resultant limits with the Poisson limits. However the mean and variance of Y may be computed and confidence intervals for values of Y reasonably near EY may be compared to the Poisson limits with respect to width and centering. It appears that when information must be thrown away to use the randomization approach, the Poisson interval based on parametric bounds may be appreciably shorter. In the optimum case for the randomization method, the interval seems to be slightly shorter on the average than the corresponding Poisson interval, at the price, however, of an increased variability.

If in the same illustration, $n_1 = 500$, $n_2 = 250$, $n_3 = 300$, $n_4 = 500$, and $X_1 = X_2 = X_3 = X_4 = 0$, then for

$$\xi_2 < \sum_{i=1}^{3} n_i - 3\min_{i=1,2,3} n_i, \text{ that is, } \xi_2 < 300,$$

one can use

$$1-\xi_1/250 \leq P \leq 1-(\xi_1/500)^2 \quad \text{or}$$

$$(1-\xi_2/250)(1-q_4^{\,2}) \leq P \leq (1-\xi_2/500)(1-q_4^{\,2}),$$

where

$$\xi_1 = \sum_{i=1}^{4} n_i q_i, \quad \xi_2 = \sum_{i=1}^{3} n_i q_i.$$

If a .025-level lower bound for P is desired, one can use the upper bound in the Poisson equal-tails .05 confidence limits for ξ_1, ξ_2. For all $X_i = 0$, the lower bounds for P turn out to be (using the $.0^41$ upper Poisson bound for q_4) .9852 and .9847 respectively. In this system, the first inequality will provide lower limits when $X_4 = 0$, and the second, when X_4 is positive. As remarked before, it is undesirable to select the confidence bounds on the basis of the particular sample point observed, and hence the greater lower parametric

bound attained by the second interval (for $q_4 < .7$) seems to make it preferable.

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In the series case, with k parts, the minimum and maximum of P (as a function of q_i's) is found. In the general case, that is, for a system having a mixture of series and parallel-connected parts, parametric minima and maxima for P are found using several approaches.

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